Final Exam Solutions
MTH 989 Representation Theory

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1 Problem 1

Lemma 1.1. *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ be a partition of $d \in \mathbb{Z}^+$, with associated Young diagram (no trailing zeros). Define $\ell_i = \lambda_i - i + k$. Then the hook lengths of the boxes in the first column are $\ell_1, \ldots, \ell_k$.*

Proof. Note that there are k boxes in the first column, since the length of λ is k. There are λ_i boxes in the ith row, and $k - i$ boxes below the ith box in the first column, so in total there are $\lambda_i + k - i = \ell_i$ boxes in the hook which has its corner at the ith box in the first row (counting down). □

Lemma 1.2 (Base case)**.** *Let $\lambda = (1 \geq 1 \geq \ldots \geq 1)$ be the "alternating" partition of $d \in \mathbb{Z}^+$. The Hook Lenght Formula holds for this particular λ.*

$$\dim V_\lambda = \frac{d!}{\prod_\lambda \text{Hook Lengths}}$$

Proof. Note that $\ell_i = 1 + d - i$. By the Frobenius formula for dimension of V_λ,

$$\dim V_\lambda = \frac{d!}{\ell_1! \ldots \ell_d!} \prod_{1 \leq i < j \leq d} (\ell_i - \ell_j) = \frac{d!}{d!(d-1)! \ldots (1)!} \prod_{1 \leq i < j \leq d} (j - i)$$

This turns out to be one, since

$$\prod_{1 \leq i < j \leq d} = (d-1)(d-2) \ldots (1)((d-1)-1)(d-1)-2) \ldots (1) \ldots = (d-1)!(d-2)! \ldots (1)!$$

Thus $\dim V_\lambda = 1$. By the previous lemma, the hook lengths for λ are $d, d-1, \ldots, 1$, so

$$\frac{d!}{\prod_\lambda \text{Hook Lengths}} = \frac{d!}{d!} = 1 = \dim V_\lambda$$

□

Proposition 1.3 (Hook Length Formula). *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ be a partition of $d \in \mathbb{Z}^+$. Then*

$$\dim V_\lambda = \frac{d!}{\prod_\lambda \text{Hook Lengths}}$$

Proof. Let $\text{col}(\lambda)$ be the number of columns of λ. We will induct on $\text{col}(\lambda)$. The case $\text{col}(\lambda) = 1$ is proven in the previous lemma. Suppose the Hook Length Formula holds for partitions with $\text{col} \leq n-1$, and let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ be a partition with $\text{col} = n$. Let $\ell_i = \lambda_i - i + k$. Let $F(\lambda)$ be the partition of $d - k$ formed by truncating the leftmost column of the Young diagram of λ. Explicitly,

$$F(\lambda) = (\mu_1 \geq \ldots \geq \mu_m) = (\lambda_1 - 1 \geq \lambda_2 - 1 \geq \ldots \geq \lambda_m - 1)$$

where $\lambda_{m+1} = \ldots = \lambda_k = 1$. Let $p_i = \mu_i + m - i$, so $p_i = \lambda_i + m - i - 1$. Let $\alpha_{ij} = \ell_i - \ell_j$, and observe that

$$\alpha_{ij} = \ell_i - \ell_j = p_i - p_j = \lambda_i - \lambda_j + j - i$$

Notice that truncating the first column does not affect the hook lengths, that is, the hook lengths of $F(\lambda)$ are the same as those for the corresponding boxes viewing the Young diagram of $F(\lambda)$ as a subdiagram of the Young diagram of λ. By our first lemma, the hook lengths of the leftmost column of λ are $\ell_1, \ldots, \ell_k$, so

$$\prod_\lambda \text{Hook Lengths} = \ell_1 \ldots \ell_k \prod_{F(\lambda)} \text{Hook Lengths} \tag{1.1}$$

By inductive hypothesis,

$$\dim V_{F(\lambda)} = \frac{(d-k)!}{\prod_{F(\lambda)} \text{Hook Lengths}} \tag{1.2}$$

By the Frobenius formula applied to $F(\lambda)$, we get

$$\dim V_{F(\lambda)} = \frac{(d-k)!}{p_1! \ldots p_m!} \prod_{1 \leq i < j \leq m} \alpha_{ij} \tag{1.3}$$

Our goal is to show that

$$\dim V_\lambda = \frac{d!}{\prod_\lambda \text{Hook Lengths}} \tag{1.4}$$

We will show that equation (1.4) is equivalent to a number of other equations, and finally prove one of these other equations holds. Combining equatiions (1.3) and (1.2), we get

$$\prod_{F(\lambda)} \text{Hook Lengths} = \frac{p_1! \ldots p_m!}{\prod_{1 \leq i < j \leq m} \alpha_{ij}} \tag{1.5}$$

Combining equations (1.1) and (1.5), we get

$$\prod_\lambda \text{Hook Lengths} = \frac{\ell_1 \ldots \ell_k p_1! \ldots p_m!}{\prod_{1 \leq i < j \leq m} \alpha_{ij}} \tag{1.6}$$

We also have the Frobenius formula for $\dim V_\lambda$, which is

$$\dim V_\lambda = \frac{d!}{\ell_1! \dots \ell_k!} \prod_{1 \le i < j \le k} \alpha_{ij} \tag{1.7}$$

As a consequence of equation (1.7), equation (1.4) is equivalent to

$$\prod_\lambda \text{Hook Lengths} = \frac{\ell_1 \dots \ell_k!}{\prod_{1 \le i < j \le k} \alpha_{ij}} \tag{1.8}$$

As a consequence of equation (1.6), equation (1.8) is equivalent to

$$\frac{\ell_1 \dots \ell_k p_1! \dots p_m!}{\prod_{1 \le i < j \le m} \alpha_{ij}} = \frac{\ell_1! \dots \ell_k!}{\prod_{1 \le i < j \le k} \alpha_{ij}} \tag{1.9}$$

which we can rearrange by cross-multiplying, applying the definition of factorial to get

$$p_1! \dots p_m! \prod_{1 \le i < j \le k} \alpha_{ij} = (\ell_1 - 1)! \dots (\ell_k - 1)! \prod_{1 \le i < j \le m} \alpha_{ij} \tag{1.10}$$

We can split this product of α_{ij} as

$$\prod_{1 \le i < j \le k} \alpha_{ij} = \left(\prod_{1 \le i < j \le m} \alpha_{ij} \right) \left(\prod_{1 \le i < j \le k, m < j} \alpha_{ij} \right) \tag{1.11}$$

which gives us that equation (1.10) is equivalent to

$$p_1! \dots p_m! \prod_{1 \le i < j \le k, m < j} \alpha_{ij} = (\ell_1 - 1)! \dots (\ell_k - 1)! \tag{1.12}$$

Simply substituting definitions for p_i, ℓ_i, α_{ij}, we write equation (1.12) in terms of λ_i.

$$\left(\prod_{i=1}^{m} (\lambda_i + m - i - 1)! \right) \left(\prod_{1 \le i < j \le k, m < j} \lambda_i - \lambda_j - i + j \right) = \prod_{i=1}^{k} (\lambda_i + i - i - 1)! \tag{1.13}$$

We divide both sides by the left term on the LHS of 1.13, which means we need the following computation.

$$\begin{aligned}
\frac{\prod_{i=1}^{k} (\lambda_i - i + k - 1)!}{\prod_{i=1}^{m} (\lambda_i - i + m - 1)!} &= \frac{\prod_{i=1}^{m} (\lambda_i - i + k - 1)!}{\prod_{i=1}^{m} (\lambda_i - i + m - 1)!} \prod_{i=m+1}^{k} (\lambda_i - i + k - 1)! \\
&= \left(\prod_{i=1}^{m} \frac{(\lambda_i - i + k - 1)!}{(\lambda_i - i + m - 1)!} \right) \prod_{i=m+1}^{k} (\lambda_i - i + k - 1)! \\
&= \left(\prod_{i=1}^{m} \prod_{\beta=1}^{k-m} \lambda_i - i + k - \beta \right) \prod_{i=m+1}^{k} (\lambda_i - i + k - 1)!
\end{aligned}$$

Thus we get the following equation, equivalent to (1.13).

$$\prod_{1\leq i<j\leq k, m<j} \lambda_i - \lambda_j - i + j = \left(\prod_{i=1}^{m}\prod_{\beta=1}^{k-m} \lambda_i - i + k - \beta\right)\left(\prod_{i=m+1}^{k} (\lambda_i = i + k - i)!\right) \quad (1.14)$$

The LHS of the above can be split into the product over $i \leq m$ and the product over $m < i$. Then we recall that $\lambda_{m+1} = \ldots = \lambda_k = 1$, which gives

$$\prod_{i=m+1}^{k} (\lambda_i - i + k - 1)! = \prod_{i=m+1}^{k} (k - i)!$$

and

$$\prod_{m<i<j\leq k} \lambda_i - 1 + j - i = \prod_{m<i<j\leq k} j - i$$

Using these, we get an equation equivalent to (1.14).

$$\left(\prod_{1\leq i\leq m<j\leq k} \lambda_i - 1 + j - i\right)\left(\prod_{m<i<j\leq k} j - i\right) = \left(\prod_{i=1}^{m}\prod_{\beta=1}^{k-m} \lambda_i - i + k - \beta\right)\left(\prod_{i=m+1}^{k} (k - i)!\right) \quad (1.15)$$

Finally, we can prove this equation holds for all $\lambda_1, \ldots, \lambda_m$. First, it is (somewhat) obvious that

$$\prod_{m<i<j\leq k} j - i = \prod_{m+1\leq i<j\leq k} j - i = \prod_{i=m+1} (k - i)! \quad (1.16)$$

Second, we claim that

$$\prod_{1\leq i<m<j\leq k} \lambda_i - i + j - 1 = \prod_{1\leq i\leq m, 1\leq\beta\leq k-m} \lambda_i - i + k - \beta \quad (1.17)$$

It is sufficient to check that the sets

$$A = \{j - 1 | m < j \leq k\} \qquad B = \{k - \beta | 1 \leq \beta \leq k - m\}$$

are equal. Since both are equal to $\{k - 1, k - 2, \ldots, m|$, we get $A = B$. Thus (1.17) holds. Since we have matched up pairs of product terms from equation (1.15), it is proven. This proves the Hook Length Formula (1.4) for λ given it for $F(\lambda)$, so this completes the induction and establishes (1.4) for all λ. As a retrospective road map, we showed that

$$(1.4) \iff (1.8) \iff (1.9) \iff (1.10) \iff (1.12) \iff (1.13) \iff (1.14) \iff (1.15)$$

and then finally proved (1.15). □

2 Problem 2

Lemma 2.1. *Let V be finite dimensional complex vector space. Let $d \in \mathbb{Z}^+$ and let $A = \{a \in \mathbb{Z} | 0 \leq a \leq d\}$. Then there exists sets $I, J \subset A$ with $I \cup J = A$, $I \cap J = \emptyset$, and*

$$\operatorname{Sym}^2(\operatorname{Sym}^d V) = \bigoplus_{a \in I} \mathbb{S}_{(d+a,d-a)} V$$
$$\wedge^2(\operatorname{Sym}^d V) = \bigoplus_{a \in J} \mathbb{S}_{(d+a,d-a)} V$$

as representations of $\mathrm{GL}(V)$.

Proof. We know that

$$\operatorname{Sym}^d V \otimes \operatorname{Sym}^d V = \operatorname{Sym}^2(\operatorname{Sym}^d V) \oplus \wedge^2(\operatorname{Sym}^d V)$$

and we know from the Pieri formula that

$$\operatorname{Sym}^d V \otimes \operatorname{Sym}^d V = \bigoplus_{a \in A} \mathbb{S}_{(d+a,d-a)} V$$

By Theorem 6.3 of Fulton and Harris, each $\mathbb{S}_{(d+a,d-a)} V$ is an irreducible representation of $\mathrm{GL}(V)$, so the above formula gives a decomposition of $\operatorname{Sym}^2(\operatorname{Sym}^d V) \oplus \wedge^2(\operatorname{Sym}^d V)$ into irreducible representations. Thus each $\mathbb{S}_{(d+a,d-a)} V$ must be a subrepresentation of exactly one of $\operatorname{Sym}^2(\operatorname{Sym}^d V)$ or $\wedge^2(\operatorname{Sym}^d V)$. The desired result is just a restatement of this in terms of a partition of the set A. □

Definition 2.1. Let W be a finite dimensional complex vector space. The **twist map** $\psi : W \otimes W$ is defined by $x \otimes y \mapsto y \otimes x$ on the spanning set $\{x \otimes y | x, y \in W\}$. Then we extend ψ by linearity. Note that if W is a representation of a group G, then ψ is equivariant, since for $g \in G$,

$$\psi g(x \otimes y) = \psi(gx \otimes gy) = gy \otimes gx = g(y \otimes x) = g\psi(x \otimes y)$$

Lemma 2.2. *Let W be a finite dimensional complex vector space. We have canonical embeddings* $\operatorname{Sym}^2(W), \wedge^2 W \hookrightarrow W \otimes W$. *Let ψ be the twist map on $W \otimes W$ defined above.*

1. *Identifying* $\operatorname{Sym}^2(W)$ *with its image under this embedding,* $\psi|_{\operatorname{Sym}^2(W)} = \mathrm{Id}$.

2. *Identifying* $\wedge^2(\operatorname{Sym}^d V)$ *with its image under this embedding,* $\psi|_{\wedge^2 W} = -\mathrm{Id}$.

Proof. Given a basis $w_1, \ldots, w_n$ of W, we have the usual induced basis for $\operatorname{Sym}^2 W$, which is $\{w_i \cdot w_j | 1 \leq i \leq j \leq n\}$, and the usual induced basis for $\wedge^2 W$, which is $\{w_i \wedge w_j | 1 \leq i < j \leq n\}$. Recall that the embeddings $\alpha : \operatorname{Sym}^2 W \hookrightarrow W^{\otimes 2}$ and $\beta : \wedge^2 \hookrightarrow W^{\otimes 2}$ are

$$\alpha(w_i \cdot w_j) = w_i \otimes w_j + w_j \otimes w_i \qquad \beta(w_i \wedge w_j) = w_i \otimes w_j - w_j \otimes w_i$$

Then we see that

$$\begin{aligned}\psi(\alpha(w_i \cdot w_j)) &= \psi(w_i \otimes w_j + w_j \otimes w_i) = \psi(w_i \otimes w_j) + \psi(w_j \otimes w_j) \\ &= w_j \otimes w_i + w_i \otimes w_j = \alpha(w_j \cdot w_i) = \alpha(w_i \cdot w_j)\end{aligned}$$

So ψ is the identity on a generating set for $\operatorname{im}\alpha$, which implies $\psi|_{\operatorname{im}\alpha} = \operatorname{Id}$. We have the analogous computation

$$\begin{aligned}\psi(\beta(w_i \wedge w_j)) &= \psi(w_i \otimes w_j - w_j \otimes w_i) = \psi(w_i \otimes w_j) - \psi(w_j \otimes w_i) \\ &= w_j \otimes w_i - w_i \otimes w_j = \beta(w_j \wedge w_i) = \beta(-w_i \wedge w_j) = -\beta(w_i \wedge w_j)\end{aligned}$$

So ψ is $-\operatorname{Id}$ on a generating set for $\operatorname{im}\beta$, which imples $\psi|_{\operatorname{im}\beta} = -\operatorname{Id}$. □

Lemma 2.3. *Let $\lambda = (\lambda_1 \geq \lambda_2)$ be a partition of $2d$. Note that $\mathbb{S}_\lambda V \subset V^{\otimes 2d}$. Let ψ be the twist map on $V^{\otimes 2d}$. Then exactly one of the following holds.*

1. *We have $\lambda_1 \equiv \lambda_2 \equiv 0 \pmod 2$ and $\psi|_{\mathbb{S}_\lambda V} = \operatorname{Id}$.*
2. *We have $\lambda_1 \equiv \lambda_2 \equiv 1 \pmod 2$ and $\psi|_{\mathbb{S}_\lambda V} = -\operatorname{Id}$.*

(NOTE: The proof of this lemma is incomplete.)

Proof. Note that since $2d$ is even and $\lambda_1 + \lambda_2 = 2d$, we must have $\lambda_1 \equiv \lambda_2 \pmod 2$, so there are no cases except for the ones in the lemma. Label λ with the standard Young tableau, so we have the boxes in the first column labelled $1, 2, \ldots, \lambda_1$ and the boxes in the second column labelled $\lambda_1 + 1, \ldots, 2d$. Let P_λ and Q_λ be the sets constructed in section 4.1 of Fulton and Harris, so

$$\begin{aligned}P_\lambda &= \{\sigma \in S_{2d} | \sigma \text{ preserves rows of } \lambda\} \\ &= \{\sigma \in S_{2d} | \sigma \text{ preserves } \{1, 2, \ldots, \lambda_1\} \text{ and } \{\lambda_1 + 1, \ldots, 2d\}\} \\ Q_\lambda &= \{\tau \in S_{2d} | \tau \text{ preserves columns of } \lambda\} \\ &= \{\tau \in S_{2d} | \tau \text{ can be written as a product of the transpositions} \\ &\qquad\qquad (1, \lambda_1 + 1), (2, \lambda_1 + 2), \ldots, (\lambda_2, 2d)\}\end{aligned}$$

Notice that if we decompose $V^{\otimes 2d}$ as $V^{\otimes \lambda_1} \otimes V^{\otimes \lambda_2}$, the action of σ on x respects this decomposition, and if we decompose as $V^{\otimes 2} \otimes V^{\otimes 2} \ldots \otimes V^{\otimes 2} \otimes V \otimes \ldots \otimes V$, then the action of τ respects this decomposition. Let a_λ, b_λ be also as defined in section 4.1.

$$a_\lambda = \sum_{\sigma \in P_\lambda} \sigma \qquad b_\lambda = \sum_{\tau \in Q_\lambda} \operatorname{sgn}(\tau)\tau$$

(Note that we are using slightly different notation for $\mathbb{C}S_{2d}$ than Fulton and Harris. Insted of e_σ, we just write σ for a basis element.) Let $c_\lambda = a_\lambda b_\lambda$. Recall how $\mathbb{C}S_{2d}$ acts on $V^{\otimes 2d}$ on the right. We have the "canonical" right action of S_{2d} on $V^{\otimes 2d}$ given by

$$V^{\otimes 2d} \times S_{2d} \to V^{\otimes 2d} \qquad (v_1 \otimes \ldots \otimes v_{2d}) \cdot \sigma = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(2d)}$$

which we then simply extend by linearity to a right $\mathbb{C}S_d$ action. Explicitly,

$$V^{\otimes 2d} \times \mathbb{C}S_{2d} \to V^{\otimes 2d} \qquad (v_1 \otimes \ldots \otimes v_{2d}) \cdot \left(\sum_i \alpha_i \sigma_i\right) = \sum_i \alpha_i (v_1 \otimes \ldots \otimes v_{2d}) \cdot \sigma_i$$

Recall that $\mathbb{S}_\lambda V$ is by definition the image of the right action of $c_\lambda \in \mathbb{C}S_{2d}$, so every element of $\mathbb{S}_\lambda V$ can be written as xc_λ where $x \in V^{\otimes 2d}$. We want to decide how ψ acts on $\mathbb{S}_\lambda V$. By linearity of everything involved,

$$\psi(xc_\lambda) = \psi(xa_\lambda b_\lambda) = \psi\left(x \sum_{\sigma \in P_\lambda} \sigma \sum_{\tau \in Q_\lambda} \operatorname{sgn}(\tau)\tau\right) = \sum_{\sigma \in P_\lambda, \tau \in Q_\lambda} \operatorname{sgn}(\tau)\psi(x\sigma\tau)$$

Since $V^{\otimes 2d}$ is generated by simple tensors (elements of the form $v_1 \otimes \ldots \otimes v_{2d}$) it is sufficient to consider only the case where x is a simple tensor. Let $x = v_1 \otimes \ldots \otimes v_{2d}$, $\sigma \in P_\lambda$, and $\tau \in Q_\lambda$. We claim that

$$\lambda_1 \equiv \lambda_2 \equiv 0 \pmod 2 \implies \psi(x\sigma\tau) = \operatorname{sgn}(\tau)x\sigma\tau$$
$$\lambda_1 \equiv \lambda_2 \equiv 1 \pmod 2 \implies \psi(x\sigma\tau) = -\operatorname{sgn}(\tau)x\sigma\tau$$

If these two claims are true, then $\psi|_{\mathbb{S}_\lambda V} = \pm \operatorname{Id}$ in the cases exactly as in the claim.

Unfortunately, I don't know how to prove this. □

Proposition 2.4. *Let V be a finite dimensional complex vector space. Then*

$$\operatorname{Sym}^2(\operatorname{Sym}^d V) = \mathbb{S}_{(2d,0)}V \oplus \mathbb{S}_{(2d-2,2)}V \oplus \ldots \oplus \ldots$$
$$\wedge^2(\operatorname{Sym}^d V) = \mathbb{S}_{(2d-1,1)}V \oplus \mathbb{S}_{(2d-3,3)}V \oplus \ldots \oplus \ldots$$

as representations of $\operatorname{GL}(V)$. *(NOTE: The proof of this depends on Lemma 2.3 which is not yet proven.)*

Proof. By Lemma 2.1, we know that each of the irreducible representations $\mathbb{S}_{(d+a,d-a)}V$ for $0 \le a \le d$ belongs to exactly one of the direct sum decompositions for $\operatorname{Sym}^2(\operatorname{Sym}^d V)$ or $\wedge^2(\operatorname{Sym}^d V)$. Thus, it is sufficient to determine which one each $\mathbb{S}_\lambda V$ fits into. By Lemma 2.2, taking $W = \operatorname{Sym}^d V$, we know that

$$\psi|_{\operatorname{Sym}^2(\operatorname{Sym}^d V)} = \operatorname{Id} \qquad \psi|_{\wedge^2(\operatorname{Sym}^d V)} = -\operatorname{Id}$$

By Lemma 2.3, we know that ψ restricts to the identity on $\mathbb{S}_{(2d,0)}V, \mathbb{S}_{(2d-2,2)}V, \ldots$, so these must be part of the decomposition of $\operatorname{Sym}^2(\operatorname{Sym}^d V)$. By the same lemma, we know that ψ restricts to the negative of the identity on $\mathbb{S}_{(2d-1,1)}V, \mathbb{S}_{(2d-3,3)}V$, so by the same logic, these must be part of the decomposition of $\wedge^2(\operatorname{Sym}^d V)$. (NOTE: In this proof, we made use of Lemma 2.3, which lacks a full proof.) □

3 Problem 3

Note: The problem as stated in the exam appears as the final proposition of this section.

Lemma 3.1. *Let L be a Lie algebra with representation W. Let $w_1, \ldots, w_n \in W$ and $x \in L$. In the action of L on $W^{\otimes n}$,*

$$x(w_1 \otimes \ldots \otimes w_n) = \sum_{i=1}^{n} v_1 \otimes \ldots \otimes x(v_i) \otimes \ldots \otimes v_n$$

(where $x(v_i)$ is the action of L on W.)

Proof. When $n = 1$, this is obvious. We induct on n. Suppose the result holds for n. Then

$$\begin{aligned} x(v_1 \otimes \ldots \otimes v_n \otimes v_{n+1}) &= x(v_1 \otimes \ldots \otimes v_n) \otimes v_{n+1} + v_1 \otimes \ldots \otimes v_n \otimes x(v_n + 1) \\ &= \left(\sum_{i=1}^{n} v_1 \otimes \ldots \otimes x(v_i) \otimes \ldots \otimes v_n \right) \otimes v_{n+1} + v_1 \otimes \ldots \otimes v_n \otimes x(v_{n+1}) \\ &= \left(\sum_{i=1}^{n} v_1 \otimes \ldots \otimes x(v_i) \otimes \ldots \otimes v_n \otimes v_{n+1} \right) + v_1 \otimes \ldots \otimes v_n \otimes x(v_{n+1}) \\ &= \sum_{i=1}^{n+1} v_1 \otimes \ldots \otimes x(v_i) \otimes \ldots \otimes v_n \otimes v_{n+1} \end{aligned}$$

This completes the induction. □

Corollary 3.2. *The above lemma holds if we replace* $W^{\otimes n}$ *with* $\operatorname{Sym}^n W$ *or* $\bigwedge^n W$.

Proof. Both $\operatorname{Sym}^n W$ and $\bigwedge^n W$ are quotients of $W^{\otimes n}$, so the action of L is the same, just composed with a quotient map. □

Lemma 3.3. *Let* $T : V \to V$ *be a linear transformation of a finite dimensional vector space* V*, with an eigenvalue* λ*. The dimension of the eigenspace for* λ *is less than or equal to the algebraic multiplicity of* λ*. More generally, the dimension of the generalized eigenspace for* λ *is equal to the algebraic multiplicity.*

Proof. Standard result from linear algebra. □

Lemma 3.4. *Let* L *be a Lie algebra with finite dimensional representation* W*. Let* $x \in L$ *be diagonalizable in the action on* W*. Viewing* $x \in \operatorname{End}(W)$*, let* $S = \{\lambda_\alpha\}_{\alpha \in A}$ *be the set of eigenvalues, each with multiplicity one. Then the eigenvalues for* $x \in \operatorname{End}(\operatorname{Sym}^n W)$ *are the multiset*

$$\left\{ \sum_{j=1}^{n} \lambda_j \,\middle|\, \lambda_j \in S \right\}$$

where sums with different ordering are considered the same. (Repeated sum totals are counted as multiplicities.)

Proof. For $\alpha \in A$, let $w_\alpha \in W$ be an eigenvector corresponding to λ_α, that is, $x(w_\alpha) = \lambda_\alpha w_\alpha$. Then by Corollary 3.2,

$$\begin{aligned} x(w_{\alpha_1} \cdot \ldots \cdot w_{\alpha_n}) &= \sum_{i=1}^{n} w_{\alpha_1} \cdot \ldots \cdot x(w_{\alpha_i}) \cdot \ldots \cdot w_{\alpha_n} \\ &= \sum_{i=1}^{n} \lambda_{\alpha_i} (w_{\alpha_1} \cdot \ldots \cdot w_{\alpha_n}) \\ &= \left(\sum_{i=1}^{n} \lambda_{\alpha_i} \right) (w_{\alpha_1} \cdot \ldots \cdot w_{\alpha_n}) \end{aligned}$$

Thus $\sum_{i=1}^{n} \lambda_{\alpha_i}$ is an eigenvalue for $x \in \operatorname{End}(\operatorname{Sym}^n V))$. Since any $w_{\alpha_1} \cdot \ldots \cdot w_{\alpha_n}$ is an element of $\operatorname{Sym}^n W$, the multiset of eigenvalues for $x \in \operatorname{End}(\operatorname{Sym}^n W))$ contains the claimed multiset. Since permutations of the order of $w_{\alpha_1} \cdot \ldots \cdot w_{\alpha_n}$ give the same element in $\operatorname{Sym}^n V$, we need to only count these sums up to reordering the λ_{α_i}.

Since $x \in \operatorname{End}(W)$ is diagonalizable, the vectors $\{W_\alpha\}$ form a basis for W, so the set of vectors of the form $w_{\alpha_1} \cdot \ldots \cdot w_{\alpha_n}$ with ascending indices form a basis for $\operatorname{Sym}^n W$. Since we have exhibited each of these as an eigenvector for $x \in \operatorname{End}(\operatorname{Sym}^n W)$, $\operatorname{Sym}^n W$ is a direct sum of eigenspaces. This means that there can't be any other eigenvalues, so the claimed multiset contains all of the eigenvalues with their multiplicities. □

Lemma 3.5. *Let ${}_2(\mathbb{C})$ have the usual Chevalley basis $\{E, F, H\}$, and let $V = \mathbb{C}^2$ be the standard representation of ${}_2(\mathbb{C})$. The multiset of eigenvalues for the action of H on $\operatorname{Sym}^n(\operatorname{Sym}^2 V))$ is*

$$\left\{ \sum_{i=1}^{n} \lambda_i \,\middle|\, \lambda_i \in \{-2, 0, 2\} \right\}$$

where the sums with different ordering of the same multiset of λ_i are considered the same.

Proof. This is a special case of the previous lemma. Take $L =_2 (\mathbb{C})$, $x = H$, and $W = \operatorname{Sym}^2 V$. Note that the eigenvalues for the action of H on $\operatorname{Sym}^2 V$ are $\{-2, 0, 2\}$. □

Definition 3.1. The **floor function** is the function $\lfloor - \rfloor : \mathbb{R} \to \mathbb{Z}$ which takes x to the greatest integer less than or equal to x.

Lemma 3.6. *Define a multiset*

$$S_n = \left\{ \sum_{i=1}^{n} \lambda_i \,\middle|\, \lambda_i \in \{-2, 0, 2\} \right\}$$

where reorderings of the λ_i do not contribute a distinct sum. Then explicitly,

$$S_n = \{2n, 2n-2, 2n-4, 2n-4, 2n-6, 2n-6, \ldots\}$$

where $2n - 2k$ has multiplicity $\lfloor k/2 \rfloor + 1$. (Incidentally, the number of elements of S_n is $\frac{(n+1)(n+2)}{2}$.)

Proof. First, notice that we have symmetry about zero, so if we enumerate the multiplicities of the terms from zero to $2n$, we are done. Consider ways that we may form a sum of n terms from $\{-2, 0, 2\}$ to form $2n - 2k$. First consider $k = 0$. There is exactly one way to form $2n$, by taking

$$2n = \sum_{i=1}^{n} 2 = 2 + \ldots + 2$$

In the case $k = 1$, there is also exactly one way to form $2n - 2$, which is by chaning exactly one 2 into a zero.

$$2n - 2 = 2 + \ldots + 2 + 0$$

We have now completed the case $n = 1$ for all k, so we may assume $n \geq 2$. For $n \geq 2$, there are $2 = \lfloor 2/2 \rfloor + 1$ possible ways to form $2n - 4$. We may replace two 2's with zeros, or we may change one to a -2.

$$2n - 4 = 2 + \ldots + 2 + \big(0 + 0\big)$$
$$2n - 4 = 2 + \ldots + 2 + \big(2 - 2\big)$$

This finishes the case $n = 2$. Similarly, for $n \geq 3$, ther e are $2 = \lfloor 3/2 \rfloor + 1$ possible ways to form $2n - 6$.

$$2n - 6 = 2 + \ldots + 2 + \big(0 + 0 + 0\big)$$
$$2n - 6 = 2 + \ldots + 2 + (0 + 2 - 2)$$

It may look like we're starting an induction, but everything we've done up to now has actually been unnecessary; it's just to get the idea across. In general, we can say that for $n \geq k$, there are $\lfloor k/2 \rfloor + 1$ possible ways to form $2n - 2k$.

$$\begin{aligned} 2n - 2k &= 2 + \ldots + 2 + (0 + \ldots + 0) \\ 2n - 2k &= 2 + \ldots + 2 + (0 + \ldots + 0 + 2 - 2) \\ 2n - 2k &= 2 + \ldots + 2 + (0 + \ldots + 0 + 2 - 2 + 2 - 2) \\ &\vdots \end{aligned}$$

We can count the ways to form $2n - 2k$ by the number of $(2 - 2)$ pairs at the end of the sum. When there are none, we get the one with k trailing zeros. Then we may insert up to $\lfloor k/2 \rfloor$ pairs $(2 - 2)$, which gives a total of $\lfloor k/2 \rfloor + 1$ different ways to form $2n - 2k$. □

Note: In the next lemma, we use the notion of multiset subtraction. If $a \in A$ has multiplicity n, and a has multiplicity m in B, then a has multiplicity $\min(n - m, 0)$ in $A \setminus B$. (We take the min with zero because we don't allow negative multiplicity in a multiset.)

Lemma 3.7. *Define the multiset*

$$S_n = \{2n, 2n - 2, 2n - 4, 2n - 4, 2n - 6, 2n - 6, \ldots, -2n + 2, -2n\}$$

where $2n - 2k$ has multiplicity $\lfloor k/2 \rfloor + 1$. Define

$$T_n = \{2n, 2n - 2, 2n - 4, \ldots, -2n\}$$

where $2n - k$ has multiplicity one. Then

$$S_n \setminus T_n = S_{n-2}$$

Proof. The multiplicity of $2n - 2k$ in $S_n \setminus T_n$ is $\lfloor k/2 \rfloor$, which is the same as $\lfloor (k-2)/2 \rfloor + 1$. □

Proposition 3.8. *Let $V = \mathbb{C}^2$ be the standard representation of $_2(\mathbb{C})$. Then*

$$\operatorname{Sym}^n(\operatorname{Sym}^2 V) = \bigoplus_{\alpha=0}^{\lfloor n/2 \rfloor} \operatorname{Sym}^{2n-4\alpha} V$$

Proof. By a previous lemmas, we know the eigenvalue multiset S_n for the action of H on $\mathrm{Sym}^n(\mathrm{Sym}^2 V)$.

$$S_n = \{2n, 2n-2, 2n-4, 2n-4, 2n-6, 2n-6, \dots, -2n+2, -2n\}$$

By the discussion in section 11.2 of Fulton and Harris, from this data we can recover the direct sum decomposition into irreducible representations of ${}_2(\mathbb{C})$, which we know all have the form $\mathrm{Sym}^k V$ for some k.

$$\mathrm{Sym}^n(\mathrm{Sym}^2 V) = \bigoplus_i \mathrm{Sym}^{k_i} V$$

In the action of H on $\mathrm{Sym}^n(\mathrm{Sym}^2 V)$, we have the eigenvalue $2n$ with multiplicity one, which means we must have one copy of $\mathrm{Sym}^{2n} V$, and can have no higher symmetric powers in our decomposition. Let W denote the remaining summands, so we can write

$$\mathrm{Sym}^n(\mathrm{Sym}^2 V) = \mathrm{Sym}^{2n} V \oplus W$$

The eigenvalue set for $\mathrm{Sym}^{2n} V$ is exactly $\{2n, 2n-2, \dots, -2n+2, -2n\}$, so the multiset of eigenvalues for H on W is $S_n \setminus T_n$, which is S_{n-2} by the above lemma. Thus by the same argument, W contains exactly one copy of$\mathrm{Sym}^{2n-4} V$ and no higher symmetric powers, so we can write W as $\mathrm{Sym}^{2n-4} V \oplus W'$.

$$\mathrm{Sym}^n(\mathrm{Sym}^2 V) = \mathrm{Sym}^{2n} V \oplus \mathrm{Sym}^{2n-4} V \oplus W'$$

Each time we obtain a summand of $\mathrm{Sym}^{2n-4\alpha}$, we subtract another T_n from S_n, so by induction we keep doing this until we exhaust all of S_n, which we will eventually do. Depending on the parity of n, this may terminate in $\mathrm{Sym}^0 V$ or $\mathrm{Sym}^2 V$.

$$\mathrm{Sym}^{2m}(\mathrm{Sym}^2 V) = \mathrm{Sym}^{4m} V \oplus \mathrm{Sym}^{4m-4} V \oplus \dots \oplus \mathrm{Sym}^0 V$$
$$\mathrm{Sym}^{2m+1}(\mathrm{Sym}^2 V) = \mathrm{Sym}^{4m+2} V \oplus \mathrm{Sym}^{4m-4} V \dots \oplus \mathrm{Sym}^2 V$$

We may write these two cases neatly in one equation, which is precisely what we claimed.

$$\mathrm{Sym}^n(\mathrm{Sym}^2 V) = \bigoplus_{\alpha=0}^{\lfloor n/2 \rfloor} \mathrm{Sym}^{2n-4\alpha} V$$

□